On universal Baxter operator for classical groups

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Abstract. The universal Baxter operator is an element of the Archimedean spherical Hecke algebra $\mathcal{H}(G,K)$, $K\subset G$ be a maximal compact subgroup of a Lie group G. It has a defining property to act in spherical principle series representations of G via multiplication on the corresponding local Archimedean L-factors. Recently such operators were introduced for $G=GL_{\ell+1}(\mathbb{R})$ as generalizations of the Baxter operators arising in the theory of quantum Toda chains. In this note we provide universal Baxter operators for classical groups $SO_{2\ell}$, $Sp_{2\ell}$ using the results of Piatetski-Shapiro and Rallis on integral representations of local Archimedean L-factors.

1 Introduction

Interactions between theory of quantum integrable systems and representation theory were intensive and mutually fruitful starting from the earlier studies of quantum integrable systems. Interpretation of the main technical tools used to solve quantum integrable systems in terms of representation theory was always a challenge promising not only a better understanding of the phenomena of quantum integrability but hopefully providing new methods in representation theory as well. One of the important results in the theory of quantum integrable systems is a construction of a class of operators introduced by Baxter [B]. The fundamental role playing by the Baxter operators in finding explicit solutions of quantum integrable systems becomes more and more obvious. However a representation theory interpretation of these operators is not quite satisfactory. Recently a progress in this direction was achieved for the Baxter operators associated with the quantum $\mathfrak{gl}_{\ell+1}$-Toda chains [GLO2]. Recall that the ring of $\mathfrak{g}$-Toda chain quantum Hamiltonians can be identified with the center $\mathcal{Z}$ of the universal enveloping algebra $\mathcal{U}\mathfrak{g}$. Common eigenfunctions of the quantum Hamiltonians are given by the $\mathfrak{g}$-Whittaker functions identified with particular matrix elements of principle series representations of $\mathcal{U}\mathfrak{g}$. In [GLO2] the integral Baxter operator for $\mathfrak{gl}_{\ell+1}$-Toda chain was introduced (see [PG] for analogous Baxter operators for affine $\widehat{\mathfrak{gl}}_{\ell+1}$-Toda chains). The integral Baxter operator is a one-parameter family of the integral operators such that the $\mathfrak{gl}_{\ell+1}$-Whittaker functions are their common eigenfunctions. Taking into account the representation theory interpretation of the quantum Hamiltonians as elements of the center $\mathcal{Z}\subset\mathcal{U}\mathfrak{g}$ it is natural to look for a similar representation theory interpretation of the Baxter operators for quantum $\mathfrak{gl}_{\ell+1}$-Toda chains. It was argued in [GLO2] that the natural representation theory framework for the Baxter operators is a theory of the spherical Hecke algebras $\mathcal{H}=\mathcal{H}(GL_{\ell+1}(\mathbb{R}),K)$ where K is a maximal compact subgroup K of $G(\mathbb{R})$. Note that the Whittaker functions can be understood as functions on $G(\mathbb{R})$ via matrix element interpretation mentioned above. Then the integral Baxter operator of $\mathfrak{gl}_{\ell+1}$-Toda chain arises via a convolution of an explicitly defined one-parameter family of elements of $\mathcal{H}$ with the $\mathfrak{gl}_{\ell+1}$-Whittaker functions. The kernels of the corresponding integral operators are given basically by the Gaussian K-biinvariant measures on $GL_{\ell+1}(\mathbb{R})$. Such family of

elements of the spherical Hecke algebra $\mathcal{H}$ was called the universal Baxter operator. A reason for this term is that such operator provides simultaneously a construction of the Baxter operators for a whole class of integrable systems associated with a pair $K \subset GL_{\ell+1}(\mathbb{R})$ (for example including along with $\mathfrak{gl}_{\ell+1}$-Toda chains also $\mathfrak{gl}_{\ell+1}$-Calogero-Sutherland systems). The surprising result of [GLO2] is that the eigenvalues of the universal Baxter operators acting on the $\mathfrak{gl}_{\ell+1}$-Whittaker functions corresponding to principle series representations $\mathcal{V}$ of $GL_{\ell+1}(\mathbb{R})$ are equal to local Archimedean L-factor attached to $\mathcal{V}$ via the local Archimedean Langlands correspondence. Such characterization of the local Archimedean L-factor is new and is in perfect correspondence with the analogous constructions over non-Archimedean fields.

In this short note we propose a construction of universal Baxter operator for classical series $Sp_{2\ell}$ and $SO_{2\ell}$ generalizing results of [GLO2]. The construction of the universal Baxter operators (Theorem 3.1) essentially relies on the results of [PSR] on integral representations for local Archimedean L-factors associated with principle series representations of $SO_{2\ell}$ and $Sp_{2\ell}$ (see [LR] for an extension to all classical series). We also show that some of the results of [PSR] can be easily rederived using the approach put forward in [GLO2]. Let us stress that the case of the classical groups other then $GL_{\ell+1}$ reveals a new phenomena. In [GLO2] we argue that the Baxter integral operators are a close cousins of the recursive operators for the Whittaker functions. Recursive operators for all classical series of finite Lie groups were constructed in [GLO1] and for classical groups other then $GL_{\ell+1}$ their integral kernels are given by non-trivial integral expressions. One should expect that a similar phenomena takes place for the universal Baxter elements of the Hecke algebra $\mathcal{H}(G,K)$ for G being classical groups other then $GL_{\ell+1}$. An explicit form of the integral kernel given in Theorem 3.1 confirm these expectations. Detailed discussion of the proposed construction of the universal Baxter operators for all classical Lie groups including a derivation of explicit integral expressions for the Baxter operators acting on Whittaker functions will appear elsewhere [GLO3].

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2 Baxter operator and spherical Hecke algebras

In this Section we review two particular classes of integral representations of $\mathfrak{gl}_{\ell+1}$-Whittaker functions. Let $E_{ij},\ i,j = 1,\dots \ell+1$ be the standard basis of the Lie algebra $\mathfrak{gl}_{\ell+1}$. Let $\mathcal{Z}(\mathcal{U}\mathfrak{gl}_{\ell+1}) \subset \mathcal{U}\mathfrak{gl}_{\ell+1}$ be a center of the universal enveloping algebra $\mathcal{U}\mathfrak{gl}_{\ell+1}$. Let $B_{\pm} \subset GL_{\ell+1}(\mathbb{C})$ be upper-triangular and lower-triangular Borel subgroups and $N_{\pm} \subset B_{\pm}$ be upper-triangular and lower-triangular unipotent subgroups. Denote by $\mathfrak{b}_{\pm} = \mathrm{Lie}(B_{\pm})$ and $\mathfrak{n}_{\pm} = \mathrm{Lie}(N_{\pm})$ their Lie algebras. Let $\mathfrak{h} \subset \mathfrak{gl}_{\ell+1}$ be a diagonal Cartan subalgebra and $W = \mathfrak{S}_{\ell+1}$ be the Weyl group of $GL_{\ell+1}$. Using the Harish-Chandra isomorphism of $\mathcal{Z}(\mathcal{U}\mathfrak{gl}_{\ell+1})$ with $\mathfrak{S}_{\ell+1}$-invariant subalgebra of the symmetric algebra $S^*\mathfrak{h}$ we identify central characters with homomorphisms $c : \mathbb{C}[h_1,\cdots,h_{\ell+1}]^{\mathfrak{S}_{\ell+1}} \to \mathbb{C}$. Let $\pi_{\underline{\lambda}} : \mathcal{U}\mathfrak{gl}_{\ell+1} \to \mathrm{End}(\mathcal{V}_{\underline{\lambda}})$, $\mathcal{V}_{\underline{\lambda}} = \mathrm{Ind}_{\mathcal{U}\mathfrak{b}_-}^{\mathcal{U}\mathfrak{gl}_{\ell+1}} \chi_{\underline{\lambda}}$ be a family of principal series representations of $\mathcal{U}\mathfrak{gl}_{\ell+1}$ induced from one-dimensional representations $\chi_{\underline{\lambda}}(b) = \prod_{j=1}^{\ell+1} |b_{jj}|^{\imath\lambda_k-\rho_k}$ of $\mathcal{U}\mathfrak{b}_-$. Here $\underline{\lambda} = (\lambda_1,\dots,\lambda_{\ell+1}) \in \mathbb{R}^{\ell+1}$ and $\rho_k = (\ell-2k+2)/2,\quad k = 1,\dots,\ell+1$. Denote $\langle\,,\,\rangle$ a pairing on $\mathcal{V}_{\underline{\lambda}}$. We suppose that the action of the Cartan subalgebra $\mathfrak{h}$ in representation $\mathcal{V}_{\underline{\lambda}}$ can be integrated to an action of the corresponding Cartan subgroup $H \subset GL_{\ell+1}(\mathbb{C})$.

The $\mathfrak{gl}_{\ell+1}$-Whittaker function can be defined as a matrix element of a principle series representation $\mathcal{V}_{\underline{\lambda}}$ of $G = GL_{\ell+1}(\mathbb{R})$ (see e.g. [J], [Ha]). Let us fix the Iwasawa decomposition $G = N_-AK-$ where K is a maximal compact subgroup of G, A be a group of diagonal matrices with the positive

elements and N_- is a maximal unipotent subgroup of lower-triangular matrices with unites on the diagonal. Let $\rho(g)$ be given by $\rho(nak) = \langle \rho, \log a\rangle$ where $k \in K$, $a \in A$ and $n \in N_-$. Let ϕ_K be a spherical vector in $\mathcal{V}_{\underline{\lambda}}$ and ψ_{N_-} be the Whittaker vector defined by the condition

$$\psi_{N_-}(bgn) = \chi_{\underline{\lambda}}(b)\chi_{N_-}(n)\,\psi_{N_-}(g), \qquad n \in N_-, \quad b \in B_-, \tag{2.1}$$

$$\phi_K(bgk) = \chi_{\underline{\lambda}}(b)\phi_K(g), \qquad\qquad k \in K,\, b \in B_-, \tag{2.2}$$

where $\chi_{N_-}(n) = \exp(2\pi\imath \sum_{j=1}^{\ell} n_{j+1,j})$. Then the Whittaker function on G is defined as a matrix element

$$\Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(g) = e^{\rho(g)}\langle \psi_{N_-}, \pi_{\underline{\lambda}}(g)\,\phi_K\rangle. \tag{2.3}$$

The function (2.3) satisfies the obvious functional equation

$$\Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(ngk) = \chi_{N_-}(n)\,\Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(g), \qquad g \in G,\, k \in K,\, n \in N_-, \tag{2.4}$$

and descends to a function on the space A of the diagonal matrices $a = \mathrm{diag}(e^{x_1}, \ldots, e^{x_{\ell+1}})$.

Standard considerations (see e.g. [STS]) show that the matrix element (2.3) is a common eigenfunction of a family of commuting differential operators descending from generators of $\mathcal{Z}(\mathcal{U}\mathfrak{gl}_{\ell+1})$ acting in $\mathcal{V}_{\underline{\lambda}}$. These differential operators can be identified with quantum Hamiltonians of $\mathfrak{gl}_{\ell+1}$-Toda chain. For example the simplest non-trivial quantum Hamiltonian acts on the Whittaker function via the differential operator

$$H_2^{\mathfrak{gl}_{\ell+1}} = -\frac{1}{2}\sum_{i=1}^{\ell+1}\frac{\partial^2}{\partial x_i^2} + 4\pi^2\sum_{i=1}^{\ell} e^{2(x_{i+1}-x_i)}.$$

Using explicit realizations of the universal enveloping algebra representation $\pi_{\underline{\lambda}}$ via difference/differential operators acting in an appropriate space of functions the matrix element representation (2.3) leads to various integral representations of the Whittaker function.

As it was demonstrated in [GLO2] the $\mathfrak{gl}_{\ell+1}$-Whittaker function being a common eigenfunction of a family of mutually commuting differential operators is also a common eigenfunction of a one-parameter family of integral operators. These integral operators were called the Baxter operators due to their relation with the Baxter operators in the theory of quantum integrable systems.

Define the Baxter Q-operator as an integral operator acting in an appropriate space of functions of $\ell+1$ variables with the integral kernel

$$\begin{aligned} Q^{\mathfrak{gl}_{\ell+1}}(\underline{x}, \underline{y}|s) = 2^{\ell+1}\, \exp\Big\{ &\sum_{j=1}^{\ell+1}(\imath s - \rho_j)(x_j - y_j) - \\ -\pi\sum_{k=1}^{\ell}\Big(e^{2(x_k-y_k)} + e^{2(y_{k+1}-x_k)}\Big) &- \pi e^{2(x_{\ell+1}-y_{\ell+1})}\Big\}, \end{aligned} \tag{2.5}$$

where $\rho = (\rho_1, \ldots, \rho_{\ell+1}) \in \mathbb{R}^{\ell+1}$, with $\rho_j = \frac{\ell}{2} + 1 - j$, $\;j = 1, \ldots, \ell+1$.

The operator $Q^{\mathfrak{gl}_{\ell+1}}(s)$ satisfies the following commutativity relations:

$$Q^{\mathfrak{gl}_{\ell+1}}(s)\cdot Q^{\mathfrak{gl}_{\ell+1}}(s') = Q^{\mathfrak{gl}_{\ell+1}}(s')\cdot Q^{\mathfrak{gl}_{\ell+1}}(s), \tag{2.6}$$

$$Q^{\mathfrak{gl}_{\ell+1}}(s)\cdot H_r^{\mathfrak{gl}_{\ell+1}} = H_r^{\mathfrak{gl}_{\ell+1}}\cdot Q^{\mathfrak{gl}_{\ell+1}}(s), \qquad r=1,\dots \ell+1, \tag{2.7}$$

where $H_r^{\mathfrak{gl}_{\ell+1}}$ are quantum Hamiltonians of $\mathfrak{gl}_{\ell+1}$-Toda chain.

With respect to $Q^{\mathfrak{gl}_{\ell+1}}(s)$ the $\mathfrak{gl}_{\ell+1}$-Whittaker function (2.3) has the following eigenfunction identity:

$$Q^{\mathfrak{gl}_{\ell+1}}(s)\,\cdot \Phi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}) := \int_{\mathbb{R}^{\ell+1}} \prod_{i=1}^{\ell+1} dy_i\; Q^{\mathfrak{gl}_{\ell+1}}(\underline{x},\,\underline{y}|\,s)\,\Phi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{y}) \;=\; L(s,\underline{\lambda})\;\Phi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}), \tag{2.8}$$

where

$$\Phi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}) = e^{-\langle\rho,x\rangle}\,\Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}). \tag{2.9}$$

We use the following notations $\underline{x}=(x_1,\dots,x_{\ell+1})$, $\underline{y}=(y_1,\dots,y_{\ell+1})$, $\underline{\lambda}=(\lambda_1,\dots,\lambda_{\ell+1})$ and the eigenvalue in (2.8) is given by

$$L(s,\underline{\lambda}) = \prod_{j=1}^{\ell+1} \pi^{-\frac{\imath s-\imath\lambda_j}{2}}\,\Gamma\Big(\frac{\imath s-\imath\lambda_j}{2}\Big). \tag{2.10}$$

The eigenvalue (2.10) is a local Archimedean L-factor associated with principle series representation $\mathcal{V}_{\underline{\lambda}}$.

The appearance of the local Archimedean L-factors in (2.8) is not accidental and is related with the fact that the integral operators (2.5) are realizations of particular elements of the local spherical Archimedean Hecke algebra (see [GLO2] for detailed discussion). Recall that the Archimedean Hecke algebra $\mathcal{H}=\mathcal{H}(G(\mathbb{R}),K)$, K being a maximal compact subgroup of $G(\mathbb{R})$ is defined as an algebra of K-biinvariant functions on $G=G(\mathbb{R})$, $\phi(g)=\phi(k_1gk_2)$, $k_1,k_2\in K$ acting by a convolution

$$\phi * f(g) = \int_G \phi(g\tilde{g}^{-1})\,f(\tilde{g})d\tilde{g} = \int_G \phi(\tilde{g})\,f(\tilde{g}^{-1}g)\,d\tilde{g}, \tag{2.11}$$

where the last equality holds for unimodular groups. To ensure the convergence of the integrals (2.11) we imply that the elements of spherical Hecke algebra belong to the Schwarz functional space i.e. the functions such all their derivatives decrease at infinity faster then inverse power of any polynomial. Spherical Hecke algebra is isomorphic to the algebra of $Ad_{G^\vee}$-invariant functions on $\mathfrak{g}^\vee=\mathrm{Lie}(G^\vee)$ where $G^\vee$ is a complex Lie group dual to G (e.g. A_ℓ, B_ℓ, C_ℓ, D_ℓ are dual to A_ℓ, C_ℓ, B_ℓ, D_ℓ respectively). Given a finite-dimensional representation $\rho_V: G^\vee\to GL(V,\mathbb{C})$ one attaches a local Archimedean L-function corresponding to a spherical irreducible representation of G as follows (see e.g. [Bu], [L]):

$$L(s',\phi,\rho_V) = \prod_{j=1}^{\ell+1} \pi^{-\frac{s'-\lambda'_j}{2}}\,\Gamma\Big(\frac{s'-\lambda'_j}{2}\Big) = \det_V\Big(\pi^{-\frac{s'-\rho_V(t_\infty)}{2}}\,\Gamma\Big(\frac{s'-\rho_V(t_\infty)}{2}\Big)\Big), \tag{2.12}$$

where $\rho_V(t_\infty)=\mathrm{diag}(\lambda'_1,\dots\lambda'_{\ell+1})$ and t_∞ is a conjugacy class in the Lie algebra $\mathfrak{g}^\vee=\mathrm{Lie}(G^\vee)$. In the case of $G=GL_{\ell+1}$, $V=\mathbb{C}^{\ell+1}$, $s'=\imath s$, $\lambda'_j=\imath\lambda_j$ we recover (2.10).

By the multiplicity one theorem [Sha], there is a unique smooth K-spherical vector ϕ_K in a principal series irreducible representation $\mathcal{V}_{\underline{\lambda}}=\mathrm{Ind}_{B_-}^G\,\chi_{\underline{\lambda}}$. The action of a K-biinvariant function

ϕ on the spherical vector ϕ_K in $\mathcal{V}_{\underline{\lambda}}$ is given by the multiplication by a character Λ_ϕ of the Hecke algebra $\mathcal{H}$:

$$\phi * \phi_K(g) = \int_G dg_1 \; \phi(gg_1^{-1}) \, \phi_K(g_1) = \Lambda_\phi(\underline{\lambda})\phi_K(g). \tag{2.13}$$

In particular, the elements ϕ of the Hecke algebra act via convolution on the Whittaker function (2.9) as follows:

$$\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}} * \phi(g) = \Lambda_\phi(\underline{\lambda}) \; \Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g), \qquad \phi \in \mathcal{H}. \tag{2.14}$$

Here the Whittaker function $\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}$ is considered as a function on G such that

$$\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(nak) = \chi_{N_-}(n) \, \Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(a), \tag{2.15}$$

where $\chi_{N_-}(n)$ is a character of N_- trivial on $[N_-, N_-]$ and $nak \in N_-AK$ is the Iwasawa decomposition of G.

One can express the eigenvalue $\Lambda_\phi(\underline{\lambda})$ corresponding to the action (2.14) of an arbitrary element $\phi \in \mathcal{H}$ as follows. Consider an action of ϕ on a normalized spherical function $\varphi_{\underline{\lambda}}(g)$ in a principle series representation $\mathcal{V}_{\underline{\lambda}}$ i.e. $\varphi_{\underline{\lambda}}$ is a matrix element in $\mathcal{V}_{\underline{\lambda}}$ such that

$$\varphi_{\underline{\lambda}}(k_1 \, g \, k_2) = \varphi_{\underline{\lambda}}(g), \qquad \varphi_{\underline{\lambda}}(e) = 1, \qquad g \in G, \quad k_1, k_2 \in K.$$

An explicit integral representation for the normalized spherical function $\varphi_{\underline{\lambda}}(g)$ is given by

$$\varphi_{\underline{\lambda}}(g) = \int_K dk \; e^{\langle h(kg), \imath\underline{\lambda} - \rho \rangle}, \tag{2.16}$$

where $h(g) = \log a$, $g = nak \in N_-AK$ is the Iwasawa decomposition of G and we normalize the volume of the compact subgroup as $\int_K dk = 1$. For an eigenvalue Λ_ϕ of ϕ

$$\varphi_{\underline{\lambda}} * \phi(g) = \; \Lambda_\phi(\underline{\lambda}) \; \varphi_{\underline{\lambda}}(g),$$

we then obtain the integral representation

$$\Lambda_\phi(\underline{\lambda}) = \; \varphi_{\underline{\lambda}} * \phi(e) = \int_G dg \phi(g^{-1}) \varphi_{\underline{\lambda}}(g). \tag{2.17}$$

In [GLO2] it was shown that the local Archimedean L-factor can be understand as an eigenvalue of a particular one-parameter family of elements $\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(s)$ of $\mathcal{H}(GL_{\ell+1}(\mathbb{R}), O_{\ell+1})$ acting on $\mathfrak{gl}_{\ell+1}$-Whittaker functions

$$\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}} * \mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(s)(g) = \prod_{j=1}^{\ell+1} \pi^{-\frac{\imath s - \imath \lambda_j}{2}} \, \Gamma\Big(\frac{\imath s - \imath \lambda_j}{2}\Big) \; \Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g), \tag{2.18}$$

and the action ϕ to the subspace of functions satisfying (2.15) coincides with the action of the integral operator $\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(s)$ given explicitly by (2.5). The corresponding one-parameter family was called universal Baxter operator.

Theorem 2.1 (GLO) *Let $\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(s)$ be a K-biinvariant function on $G=GL_{\ell+1}$ given by*

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(g,s)=2^{\ell+1}\,|\det g|^{\imath s+\frac{\ell}{2}}e^{-\pi \mathrm{Tr} g^t g}. \tag{2.19}$$

Then the action of $\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(s)$ on the Whittaker function $\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g)$ by a convolution descends to the action of $Q^{\mathfrak{gl}_{\ell+1}}(s)$ with the integral kernel (2.5) *and satisfies the relation*

$$\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}} * \mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(s)(g)=L(s,\underline{\lambda})\;\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g), \tag{2.20}$$

where $L(s,\underline{\lambda})$ is the local Archimedean L-factor

$$L(s,\underline{\lambda})=\prod_{j=1}^{\ell+1}\pi^{-\frac{\imath s-\imath\lambda_j}{2}}\Gamma\Big(\frac{\imath s-\imath\lambda_j}{2}\Big). \tag{2.21}$$

Below we prove a slightly modified version of this Theorem.

Theorem 2.2 *Let $\tilde{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(s)$ be a K-biinvariant function on $G=GL_{\ell+1}$ given by*

$$\widetilde{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(g,s):=\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(g^{-1},s)=2^{\ell+1}\,|\det g^{-1}|^{\imath s+\frac{\ell}{2}}e^{-\pi \mathrm{Tr}(g^{-1})^t g^{-1}}. \tag{2.22}$$

Then the action of $\tilde{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(g,s)$ on the modified Whittaker function $\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g)$ by a convolution descends to the action of $\tilde{Q}^{\mathfrak{gl}_{\ell+1}}(s)$ with the integral kernel

$$\tilde{Q}^{\mathfrak{gl}_{\ell+1}}(\underline{x},\underline{y}|s)=2^{\ell+1}\;\exp\Big\{\sum_{j=1}^{\ell+1}(\imath s+\rho_j)(y_j-x_j)-$$

$$-\pi\sum_{k=1}^{\ell}\Big(e^{2(y_k-x_k)}+e^{2(x_{k+1}-y_k)}\Big)-\pi e^{2(y_{\ell+1}-x_{\ell+1})}\Big\},$$

and satisfies the relation

$$\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}} * \widetilde{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(s)(g)=L(s,-\underline{\lambda})\;\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g), \tag{2.23}$$

where $L(s,\underline{\lambda})$ is the local Archimedean L-factor given by (2.21).

Proof. The convolution of a K-biinvariant function with $\mathfrak{gl}_{\ell+1}$-Whittaker functions is given by

$$\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}} * \tilde{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(g)=\int_G d\tilde{g}\;\tilde{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(g\tilde{g}^{-1})\,\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\tilde{g})=\int_G d\tilde{g}\;\tilde{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(g\tilde{g}^{-1})\,\langle\psi_{N_-},\pi_{\underline{\lambda}}(\tilde{g})\phi_K\rangle. \tag{2.24}$$

Fix the Iwasawa decomposition $\tilde{g}=\tilde{n}\tilde{a}\tilde{k}$, $\tilde{k}\in K$, $\tilde{a}\in A$, $\tilde{n}\in N_-$ of a generic element $\tilde{g}\in G$ and let $\delta_{B_-}(\tilde{a})=\det_{\mathfrak{n}_-}\mathrm{Ad}_{\tilde{a}}$. We normalize the volume of the compact subgroup as $\int_K dk=1$. We shall use the notation $d^\times a=da\cdot\det(a)^{-1}$ for $a\in A$. Then for $a\in A$ we have

$$\begin{aligned}\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}} * \tilde{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(a)&=\int_{AN_-}d^\times\tilde{a}d\tilde{n}\,\delta_{B_-}(\tilde{a})\,\tilde{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(a\tilde{n}^{-1}\tilde{a}^{-1})\,\chi_{N_-}(\tilde{n})\,\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\tilde{a})\;=\\&=\int_A d^\times\tilde{a}\;K_{\tilde{\phi}}(a,\tilde{a})\,\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\tilde{a}),\end{aligned} \tag{2.25}$$

with

$$K_{\tilde{\phi}}(a,\tilde{a}) = \int_{N_-} d\tilde{n}\ \delta_{B_-}(\tilde{a})\ \mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\tilde{a}\tilde{n}a^{-1})\,\chi_{N_-}(\tilde{n}),$$

$$\chi_{N_-}(\tilde{n}) = \exp\Big\{\,2\pi\imath\sum_{i=1}^{\ell}\tilde{n}_{i+1,i}\Big\}.$$

Thus to prove the expression (2.23) for the integral kernel we should prove the following

$$\tilde{Q}^{\mathfrak{gl}_{\ell+1}}(\underline{x},\underline{y}|s) = \int_{N_-} d\tilde{n}\ \delta_{B_-}(\tilde{a})\,\mathcal{Q}(\tilde{a}\tilde{n}a^{-1}|s)\,\chi_{N_-}(\tilde{n}), \tag{2.26}$$

where

$$a = \ \mathrm{diag}\,(e^{x_1},\ldots,e^{x_{\ell+1}}), \qquad \tilde{a} = \mathrm{diag}(e^{y_1},\ldots,e^{y_{\ell+1}}),$$
$$\delta_{B_-}(\tilde{a}) = e^{2\langle\rho,\log\tilde{a}\rangle} = e^{\sum_{i<j}(y_i-y_j)}. \tag{2.27}$$

For $g=\tilde{a}\tilde{n}a^{-1}$ we have

$$\det\,g = e^{\sum_{i=1}^{\ell+1}(y_i-x_i)}, \qquad \mathrm{Tr}\ g^tg = \sum_{i=1}^{\ell+1} e^{2(y_i-x_i)} + \sum_{i<j}\tilde{n}_{ij}^2e^{2(y_i-x_j)}, \tag{2.28}$$

where $\tilde{n}\in N_-$. Taking into account that $\chi_{N_-}(\tilde{n}) = \exp(2\pi\imath\sum_{i=1}^{\ell}\tilde{n}_{i+1,i})$ we obtain

$$\tilde{Q}^{\mathfrak{gl}_{\ell+1}}(\underline{x},\underline{y}|s) = 2^{\ell+1}\int_{N_-} d\tilde{n}\ e^{\sum_{i<j}(y_i-y_j)}\,e^{2\pi\imath\sum_{k=1}^{\ell}\tilde{n}_{i+1,i}}\ \times \tag{2.29}$$

$$\exp\Big\{\sum_{i=1}^{\ell+1}(\imath s+\frac{\ell}{2})(y_i-x_i) - \pi\sum_{i=1}^{\ell+1}e^{2(y_i-x_i)} - \pi\sum_{i<j}\tilde{n}_{ij}^2e^{2(y_i-x_j)}\Big\} =$$

$$2^{\ell+1}\ \exp\Big\{(\imath s+\frac{\ell}{2})\sum_{i=1}^{\ell+1}(y_i-x_i) - \pi\sum_{i=1}^{\ell+1}e^{2(y_i-x_i)}\Big\}\,e^{\sum_{i<j}(y_i-y_j)}\ \times \tag{2.30}$$

$$\int_{\mathbb{R}^\ell}\prod_{i=1}^{\ell}d\tilde{n}_{i+1,i}\ \exp\Big\{2\pi\imath\sum_{k=1}^{\ell}\tilde{n}_{i+1,i} - \pi\sum_{i=1}^{\ell}\tilde{n}_{i+1,i}^2e^{2(y_i-x_{i+1})}\Big\}\ \times$$

$$\prod_{i>j+1}\int d\tilde{n}_{ij}\ \exp\Big\{-\pi\tilde{n}_{ij}^2e^{2(y_j-x_i)}\Big\}.$$

Computing the integrals by using the formula

$$\int_{-\infty}^{\infty}e^{\imath\omega x-px^2}dx = \sqrt{\frac{\pi}{p}}\ \ e^{-\frac{\omega^2}{4p}}, \qquad p>0, \tag{2.31}$$

we readily obtain

$$\tilde{Q}^{\mathfrak{gl}_{\ell+1}}(\underline{x},\underline{y}|s) = 2^{\ell+1}\ \exp\Big\{\sum_{i=1}^{\ell+1}(\imath s+\rho_i)(y_i-x_i) - \tag{2.32}$$

$$-\pi\sum_{i=1}^{\ell}\Big(e^{2(y_i-x_i)}+e^{2(x_{i+1}-y_i)}\Big) - \pi e^{2(y_{\ell+1}-x_{\ell+1})}\Big\},$$

where $\rho_j = \frac{\ell}{2}+1-j,\ \ j=1,\dots,\ell+1$. This completes the proof of the formula (2.23) for the integral kernel.

Now we prove (2.23). Taking into account (2.17) we shall check the following relation:

$$\prod_{j=1}^{\ell+1}\pi^{-\frac{\imath s+\imath\lambda_j}{2}}\,\Gamma\left(\frac{\imath s+\imath\lambda_j}{2}\right)=\int_G dg\ \tilde{\phi}(g^{-1},s)\,\varphi_{\underline{\lambda}}(g), \tag{2.33}$$

where Im$s<0$ is assumed. The right hand side of (2.33) can be written as follows

$$2^{\ell+1}\int_{G\times K} dk\,dg\,|\det g|^{\imath s+\frac{\ell}{2}}e^{-\pi\mathrm{Tr}\,(g^tg)}\,e^{<h(kg),\imath\underline{\lambda}-\rho>}=2^{\ell+1}\int_G dg\,|\det g|^{\imath s+\frac{\ell}{2}}e^{-\pi\mathrm{Tr}\,(g^tg)}\,e^{<h(g),\imath\underline{\lambda}-\rho>}$$

$$=2^{\ell+1}\int_{K\times A\times N_-} dn\,d^\times a\,dk\,\delta_{B_-}(a)|\det a|^{\imath s+\frac{\ell}{2}}e^{-\pi\mathrm{Tr}\,(na^2n^t)}\,e^{\langle\log(a),\imath\underline{\lambda}-\rho\rangle}$$

$$=2^{\ell+1}\int_{A\times N_-} dn\,d^\times a\,\delta_{B_-}(a)|\det a|^{\imath s+\frac{\ell}{2}}e^{-\pi\mathrm{Tr}\,(na^2n^t)}\,e^{<\log(a),\imath\underline{\lambda}-\rho>}$$

$$=\prod_{j=1}^{\ell+1}\pi^{-\frac{\imath s-\imath\lambda_j}{2}}\,\Gamma(\frac{\imath s+\imath\lambda_j}{2}),$$

where the following formula was used

$$\int_{-\infty}^{+\infty}dxe^{\nu x}e^{-ae^{2x}}=\frac{1}{2}a^{-\frac{\nu}{2}}\,\Gamma(\frac{\nu}{2}),\qquad \mathrm{Re}\,\nu>0,\quad a>0. \tag{2.34}$$

□

Combining Theorems 2.1, 2.2 one obtains the following result proved in [PSR].

Proposition 2.1 *Let $\mathcal{Q}^{(2)}$ be a K-biinvariant function on $GL_{\ell+1}(\mathbb{R})$ as given by*

$$\mathcal{Q}^{(2)}(g^{-1},s)=\,|\det g|^{\imath s+\ell/2}\int_{GL_{\ell+1}(\mathbb{R})}e^{-\pi\mathrm{Tr}\,Z\,(gg^t\,+1)\,Z^t}\,|\det Z|^{2\imath s+\ell}\,dZ. \tag{2.35}$$

Here dZ is the standard Haar measure on $GL_{\ell+1}(\mathbb{R})$. Then one has the identity

$$\Phi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}*\mathcal{Q}^{(2)}(g,s)=L(s,\underline{\lambda})\ L(s,-\underline{\lambda})\ \Phi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(g), \tag{2.36}$$

where $L(s,\underline{\lambda})$ is the local Archimedean L-factor (2.21).

Proof. Note that the function (2.35) can be represented as a convolution

$$\mathcal{Q}^{(2)}(g,s)=\widetilde{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}*\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(g,s),\qquad \mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(g,s)=\,|\det g|^{\imath s+\ell/2}\,e^{-\pi\mathrm{Tr}\,g^tg},\qquad \widetilde{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(g,s)=\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(g^{-1},s).$$

The statement of the Proposition directly follows from Theorems 2.1, 2.2 (taking into account that our normalization of the Haar measure differs from that in [PSR] by the factor $2^{\ell+1}$). □

3 Universal Baxter operator for classical groups $Sp_{2\ell}$ and $SO_{2\ell}$

In this Section we propose a generalization of Theorem 2.1 to the case of the maximal split forms of the classical series $SO_{2\ell}$ and $Sp_{2\ell}$. Note that the resulting expressions for analogs of (2.19) are not as simple as in the case of general linear groups and are given by non-trivial integrals.

Let us first recall a particular realization of classical Lie groups $SO_{2\ell}$ and $Sp_{2\ell}$ as subgroups of the general linear group $GL_{2\ell}$. We are interested in realizations of standard representations $\pi_{2\ell} : \mathfrak{g} \to End(\mathbb{C}^{2\ell})$ of the Lie algebras $\mathfrak{g} = \mathfrak{sp}_{2\ell}$, $\mathfrak{so}_{2\ell}$ such that the Weyl generators corresponding to the Borel (Cartan) subalgebras of $\mathfrak{g}$ are realized by upper triangular (diagonal) matrices (see e.g. [DS]). The corresponding group embedding $G \to GL_{2\ell}$, $G = SO_{2\ell}\,, Sp_{2\ell}$ can be defined as follows. Consider the following involution on $GL_{2\ell}$ (identified with a subspace of $2\ell \times 2\ell$-matrices):

$$g \longmapsto g^* := S \cdot J \cdot (g^{-1})^t \cdot J^{-1} \cdot S^{-1}, \tag{3.1}$$

where $a \to a^t$ is induced by the standard matrix transposition,

$$S = \mathrm{diag}(1, -1, \ldots, -1, 1), \tag{3.2}$$

and $J = \|J_{i,j}\| = \|\delta_{i+j,2\ell+1}\|$. The symplectic group $G = Sp_{2\ell}$ then can be defined as the following subgroup of $GL_{2\ell}$:

$$Sp_{2\ell} = \{g \in GL_{2\ell} : g^* = g\}. \tag{3.3}$$

Similarly in the case of $G = SO_{2\ell}$ consider the involution on $GL_{2\ell}$

$$g \longmapsto g^* := S \cdot J \cdot (g^{-1})^t \cdot J^{-1} \cdot S^{-1}, \tag{3.4}$$

where

$$S = \mathrm{diag}(1, -1, \ldots, (-1)^{\ell-1}, (-1)^{\ell-1}, (-1)^{\ell}, \cdots, 1), \tag{3.5}$$

and $J = \|J_{i,j}\| = \|\delta_{i+j,2\ell+1}\|$. The orthogonal group $G = SO_{2\ell}$ can be defined as the following subgroup of $GL_{2\ell}$:

$$SO_{2\ell} = \{g \in GL_{2\ell} : g^* = g\}. \tag{3.6}$$

The maximal compact subgroup of $G = SO_{2\ell}, Sp_{2\ell}$ embedded this way is given by an intersection of G with the maximal compact subgroup of $GL_{2\ell}(\mathbb{R})$.

We would like to construct elements of Hecke algebra $\mathcal{H}(G, K)$, where G are maximal split forms of $Sp_{2\ell}$, $SO_{2\ell}$ and $K \subset G$ is a maximal compact subgroup, such that their actions on spherical vectors in spherical principle series representations are given by multiplications on the corresponding local Archimedean L-factors associated with the standard representations of the dual Lie groups. Recall that $SO_{2\ell}$ is self-dual and $Sp_{2\ell}$ is dual to $SO_{2\ell+1}$. The corresponding local L-factors are given by

$$L^{SO_{2\ell}}(s, \mu_1, r) = \prod_{i=1}^{\ell} \Gamma_{\mathbb{R}}(s, \lambda_i)\, \Gamma_{\mathbb{R}}(s, -\lambda_i), \qquad G = SO_{2\ell}, \quad G^\vee = SO_{2\ell}, \tag{3.7}$$

$$L^{SO_{2\ell+1}}(s, \mu_2, r) = \Gamma_{\mathbb{R}}(s, 0) \prod_{i=1}^{\ell} \Gamma_{\mathbb{R}}(s, \lambda_i)\, \Gamma_{\mathbb{R}}(s, -\lambda_i), \quad G = Sp_{2\ell}, \quad G^\vee = SO_{2\ell+1}, \tag{3.8}$$

where we parameterize the weights of the corresponding principle series representations via elements of the dual Lie algebras by taking $\mu_1 = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_\ell, -\lambda_1, \dots, -\lambda_\ell)$ for an element of the Lie algebra $\mathfrak{so}_{2\ell}$ and similarly $\mu_2 = \mathrm{diag}(0, \lambda_1, \lambda_2, \dots, \lambda_\ell, -\lambda_1, \dots, -\lambda_\ell)$ for an element of the Lie algebra $\mathfrak{so}_{2\ell+1}$. Here we also use the modified Γ-function:

$$\Gamma_{\mathbb{R}}(s, \lambda) = \pi^{-\frac{\imath s - \imath\lambda}{2}} \Gamma\Big(\frac{\imath s - \imath\lambda}{2}\Big). \tag{3.9}$$

The following Theorem is a simple reformulation of the result obtained in [PSR].

Theorem 3.1 *Let G be either $SO_{2\ell}$ or $Sp_{2\ell}$ and $G^\vee$ be the corresponding dual group $SO_{2\ell}$ or $SO_{2\ell+1}$. Let $\mathcal{Q}_G(g, s)$ be a one-parameter family of functions on G given by*

$$\mathcal{Q}_G(g, s) = d_G(s)\, \frac{R_G(g, s)}{R_G(0, s)}, \tag{3.10}$$

where

$$R_G(g, s) = \int_{GL_{2\ell}(\mathbb{R})} e^{-\pi \mathrm{Tr}\, Z^t (g^t g + 1) Z}\, |\det Z|^{\imath s}\, dZ, \tag{3.11}$$

$$R_G(0, s) = \int_{GL_{2\ell}(\mathbb{R})} e^{-\pi \mathrm{Tr}\, Z^t Z}\, |\det Z|^{\imath s}\, dZ,$$

and

$$d_{SO_{2\ell}}(s) = \prod_{j \equiv 0 (\mathrm{mod} 2), j=0}^{2\ell-2} \Gamma_{\mathbb{R}}(2\imath s - j),$$

$$d_{Sp_{2\ell}}(s) = \prod_{j \equiv 0 (\mathrm{mod} 2), j=2}^{2\ell} \Gamma_{\mathbb{R}}(2\imath s - j).$$

We also imply that G is embedded in $GL_{2\ell}$ via (3.3), (3.6).

Then $\mathcal{Q}_G(g, s)$ is a K-biinvariant function such that its action on a spherical vector ϕ_K in a principle series representation $\mathcal{V}_{\underline{\mu}} = \mathrm{Ind}_{B_-}^G \chi_{\underline{\mu}}$ is via multiplication on the corresponding local Archimedean L-factor (given by (3.7), (3.8)*)*

$$\phi_K * \mathcal{Q}_G(s)(g) = \int_G dg_1\, \mathcal{Q}_G(g_1, s)\, \phi_K(g g_1^{-1}) = L^{G^\vee}(s, \underline{\mu}) \phi_K(g), \tag{3.12}$$

and thus $\mathcal{Q}_G(s)$ is a universal Baxter operator for classical groups $SO_{2\ell}$, $Sp_{2\ell}$. In particular it acts on the Whittaker functions (2.15) *associated with the classical groups G by a multiplication on the corresponding local Archimedean L-factor* (3.7), (3.8).

Proof. Let $\varphi_{\underline{\mu}}(g)$ be a normalized spherical function on G corresponding to the principle series representation $\mathcal{V}_{\underline{\mu}}$. According to [PSR] (see e.g. p. 37) the following integral representations for Archimedean L-factors (3.7), (3.8) holds

$$L^{G^\vee}(s, \mu) = \int_G \mathcal{Q}_G(g, s)\, \varphi_{\underline{\mu}}(g^{-1}) dg, \tag{3.13}$$

where $\mathcal{Q}_G(g, s)$ is given by (3.10). Now using (2.13) and (2.17) we obtain (3.12). □

As a simple illustration of the above results let us provide explicit calculations for $\ell = 1$. The maximal split form of SO_2 can be embedded in $GL_2(\mathbb{R})$ via diagonal matrices. Thus the connected component of the unity can be parametrized as follows:

$$g = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

This is in agreement with (3.6) with $S = \mathrm{diag}(1,1)$. The normalized spherical function is given by

$$\varphi_{\underline{\mu}}(g^{-1}(t)) = e^{-\imath\lambda t}, \qquad \underline{\mu} = (\lambda, -\lambda).$$

Specialization of (3.11) to $G = SO_2$ gives

$$R_{SO_2}(g,s) = \int_{GL_2(\mathbb{R})} e^{-\pi \mathrm{Tr}\, Z^t(g^t g+1)Z} \,|\det Z|^{\imath s}\, dZ.$$

To calculate the integral let us denote $X^2 = g^t g + 1 = \mathrm{diag}(X_1^2, X_2^2)$ and $X_1^2 = e^{2t}+1$, $X_2^2 = e^{-2t}+1$. Then we have

$$\mathrm{Tr}\, Z^t X^2 Z = \sum_{i,j=1}^{2} Z_{ij}^2 X_i^2.$$

Using the change of variables $Z_{ij} \to Z_{ij} X_i^{-1}$ we obtain

$$R_{SO_2)}(g,s) = |X_1 X_2|^{-\imath s} \int_{GL_2(\mathbb{R})} e^{-\pi \mathrm{Tr}\, Z^t Z} \,|\det Z|^{\imath s}\, dZ.$$

Thus we have

$$\mathcal{Q}_{SO_2}(g,s) = d_{SO_2}(\imath s)\, \frac{R_{SO_2}(g,s)}{R_{SO_2}(0,s)} = \frac{\Gamma_{\mathbb{R}}(2\imath s)}{(e^{2t}+1)(1+e^{-2t})^{\imath s/2}} = \frac{\Gamma_{\mathbb{R}}(2\imath s)}{(e^t+e^{-t})^{\imath s}}. \tag{3.14}$$

Now we would like to calculate the integral

$$L^{SO_2}(s,\underline{\mu}) = 2\int_{\mathbb{R}} dt\, e^{-\imath\lambda t}\, \mathcal{Q}_{SO_2}(g(t),s),$$

where prefactor 2 takes into account sum over connected components of the split form of $SO(2)$. Taking into account (3.14) and using the Euler integral representation for the Gamma-function we have (assuming $\mathrm{Im} s < 0$)

$$L^{SO_2}(s,\underline{\mu}) = 2\int_{\mathbb{R}^2} d\tau\, dt\, e^{\imath s\tau} e^{-\imath\lambda t} e^{-\pi e^{\tau}(e^t+e^{-t})}$$

$$= \int_{\mathbb{R}^2} dt_1\, dt_2\; e^{t_1 \frac{(\imath s - \imath\lambda)}{2} + t_2 \frac{(\imath s + \imath\lambda)}{2}} e^{-\pi(e^{t_1}+e^{t_2})} = \pi^{-\frac{\imath s - \imath\lambda}{2}} \Gamma\left(\frac{\imath s - \imath\lambda}{2}\right) \pi^{-\frac{\imath s + \imath\lambda}{2}} \Gamma\left(\frac{\imath s + \imath\lambda}{2}\right) = L_{SO_2}(s,\mu).$$

One can directly calculate the action on SO_2-Whittaker functions

$$\Phi_{\underline{\mu}}(x) = e^{\imath\lambda x}, \qquad \underline{\mu} = (\lambda, -\lambda).$$

We have

$$\Phi_{\underline{\mu}} * \mathcal{Q}_{SO_2}(s)(x) = \pi^{-\frac{\imath s - \imath\lambda}{2}} \Gamma\left(\frac{\imath s - \imath\lambda}{2}\right) \pi^{-\frac{\imath s + \imath\lambda}{2}} \Gamma\left(\frac{\imath s + \imath\lambda}{2}\right) \Phi_{\underline{\mu}}(x) = L^{SO(2)}(s,\underline{\mu}) \Phi_{\underline{\mu}}(x).$$

This completes explicit verification of the statement of Theorem 3.1 for $G = SO_2$.

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